Module V

Graph Representation and Vertex colouring.

- Matrix representation of graphs
- Adjacency matrix
- Incidence matrix
- Circuit matrix
- Path matrix
- Coloring
- chromatic number
- chromatic polynomial
- Matchings
- Coverings
- Four colour Problem
- Five colour Problem
- Greedy colouring algorithm.

Matrix representation of Graphs

$A(G)$

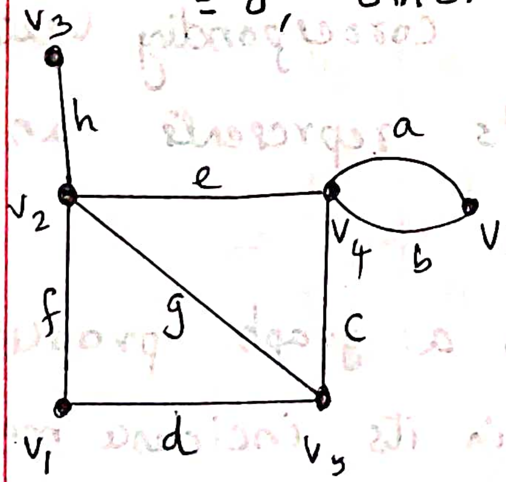
Incidence Matrix: [vertex-edge incidence matrix]

Let G be a graph with n vertices e edges and no self loops. Define an n by e matrix $A = [a_{ij}]$ whose n rows corresponds to the n vertices and the e columns corresponds to the e edges as follows :

The matrix element,

$a_{ij} = 1$, if j^{th} edge e_j is incident on i^{th} vertex v_i

$= 0$, otherwise



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{bmatrix} a & b & c & d & e & f & g & h \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The incidence matrix of G is writen as A(G).

Note:

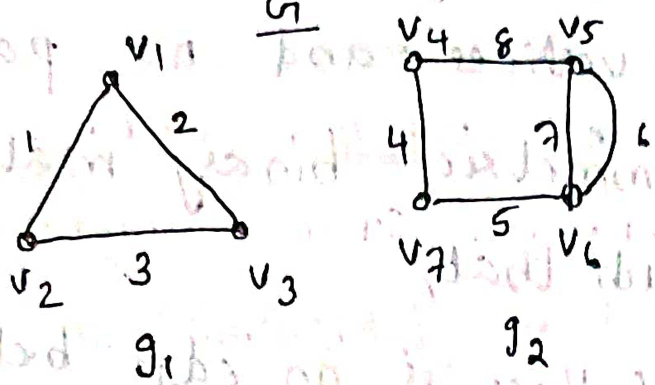
The incidence matrix contains only two elements 0 and 1. Such a matrix is called binary matrix or (0,1)- matrix

Remarks:

- Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.
- The number of 1's in each row equals the degree of the corresponding vertex
- A row with all 0's represents an isolated vertex
- Parallel edges in a graph produces identical columns in its incidence matrix
- If a graph G is disconnected and consists of two components g_1 and g_2, A(G) ca be written in a block-diagonal form

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]$$ where $A(g_1)$

and $A(g_2)$ are the incidence matrices of components g_1 and g_2.



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{ccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]$$

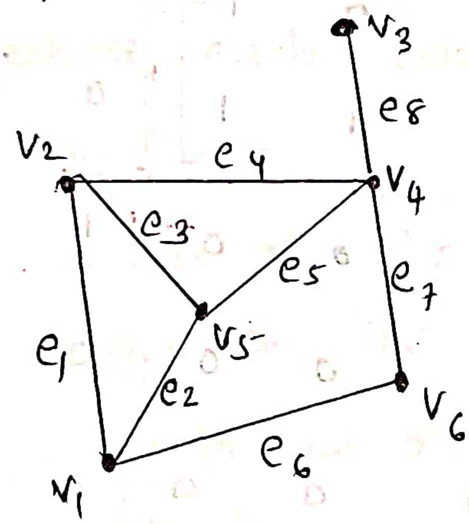
Adjacency Matrix $X(G)$

The adjacency matrix of a graph G with n vertices and no parallel edges is an $n \times n$ symmetric binary matrix $X = [x_{ij}]$ such that,

$x_{ij} = 1$, if there is an edge between i^{th} and j^{th} vertices

$= 0$, if there is no edge between them.

Eg.



$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Remarks:

- The entries along the principal diagonal of the matrix X are all 0's if and only if the graph has no self loops.

- The adjacency matrix has no provision for parallel edges.

- If the graph has no self loops the degree of a vertex equals the number of 1's in the corresponding row or column of X.

- A graph G is disconnected and its two components g_1 and g_2 iff its adjacency matrix $X(G)$ can be partitioned as

$$X(G) = \begin{bmatrix} X(g_1) & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & X(g_2) \end{bmatrix}$$

$X(g_1)$ & $X(g_2)$ are adjacency matrix of g_1 & g_2 respectively.

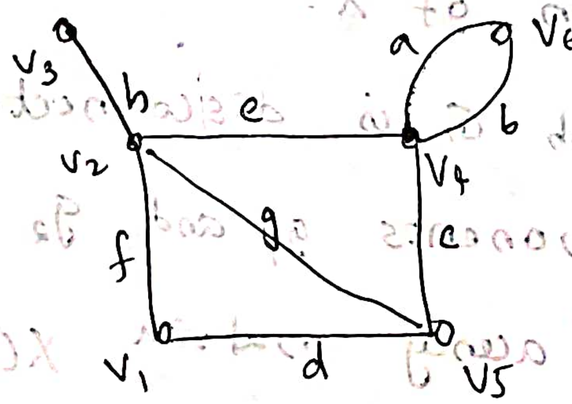
Circuit Matrix $B(G)$

Let the number of different circuits in a graph G be q and the number of edges in G be e. Then the circuit matrix $B = [b_{ij}]$ of G is a q by e, $(0,1)$-matrix defined as

$b_{ij} = 1$, if ith circuit includes jth edge
$= 0$, otherwise

Circuit matrix is denoted as $B(G)$

Eg:



The above graph has 4 different circuits: $\overset{1}{\{a, b\}}$, $\overset{2}{\{c, e, g\}}$, $\overset{3}{\{d, f, g\}}$ and $\overset{4}{\{c, d, f, e\}}$

circuit matrix is a 4 by 8 $(0,1)$-matrix given by

$$B(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Remarks:

- A column of all zeros corresponds to a noncircuit edge.
- Each row of $B(G)$ is a circuit vector
- A selfloop is possible
- The number of 1's in the a row is equal to the number of edges in the corresponding circuits.
- If G is disconnected and consists of components g_1 & g_2 $B(G)$ can be written in

$$\begin{bmatrix} B(g_1) & 0 \\ 0 & B(g_2) \end{bmatrix}$$

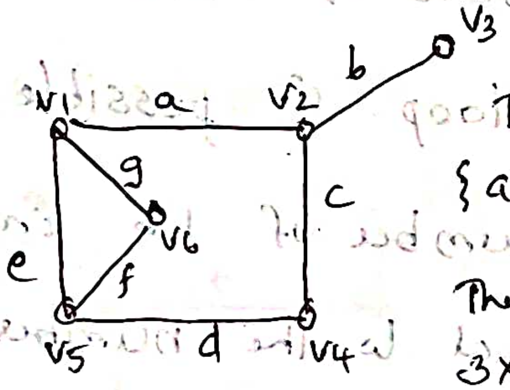
where $B(g_1)$ & $B(g_2)$ are circuits matrices of g_1 & g_2

Path Matrix

A path matrix is defined for a specific pair of vertices in a graph say (x, y). and is written as $P(x, y)$. i.e. the path matrix for (x, y) vertices is $P(x, y) = [P_{ij}]$ where,

$P_{ij} = 1$, if j^{th} edge lies in i^{th} path and
$= 0$, otherwise

eg:



Consider all paths between v_1 & v_3.

There are 3 different paths:

$\{a, b\}$, $\{e, d, c, b\}$, $\{g, f, d, c, b\}$
1, 2, 3

Then we get the following 3×7 path matrix.

$$P(v_1, v_3) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Remarks:

- A column of all 0's corresponds to an edge that does not lie in any path between x & y.
- A column of all 1's corresponds to an edge that lies in every path between x and y.
- There is no row with all 0's
- The ring sum of any two rows in $P(x, y)$ corresponds to a circuit or an edge disjoint union of circuits

Coloring (proper coloring)

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same colour is called the proper coloring of a graph.

A graph in which every vertex has been assigned a color according to a proper coloring is called a properly coloured graph.

Chromatic number of a graph $X(G)$

A graph that requires k different colours for its proper coloring and no less is called a k chromatic graph $X(G)$ and the number k is called the chromatic number of the graph G.

Remarks:

- A graph consisting of only isolated vertices is 1-chromatic

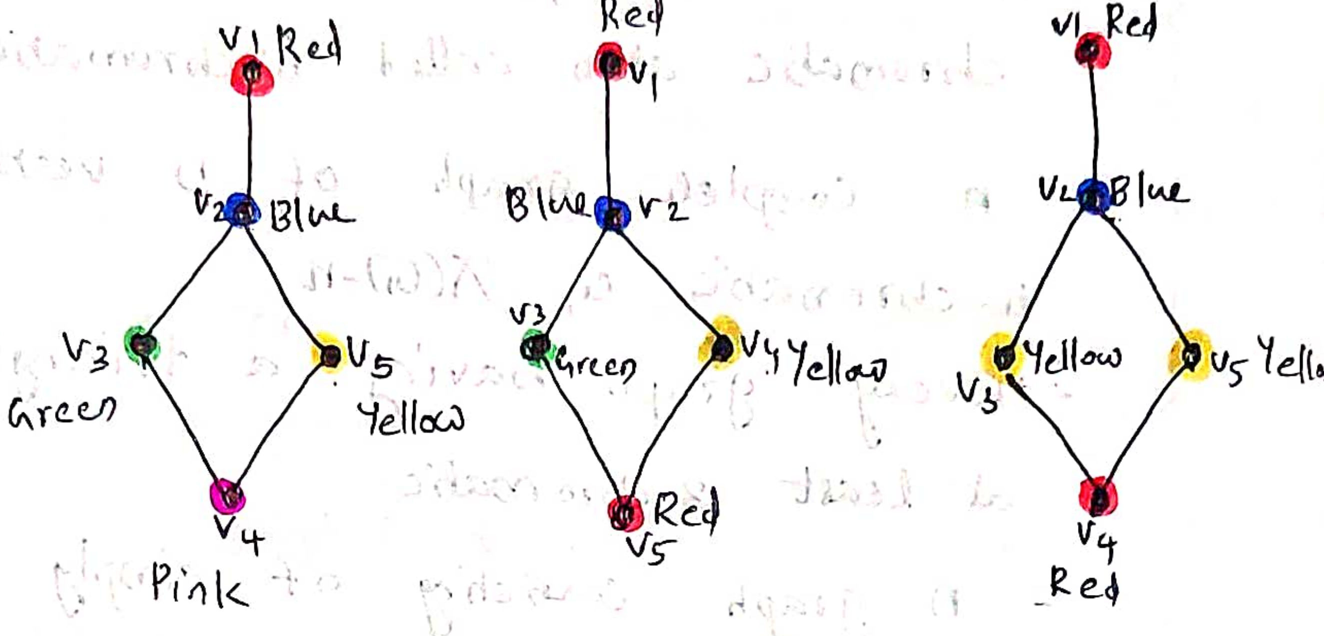
- A graph with one or more edges (not a self loop) is at least 2-chromatic also called bichromatic.

- A complete graph of n vertices is n-chromatic i.e. $\chi(G)=n$

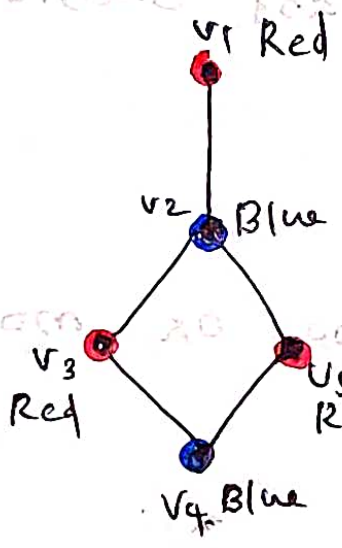
- Every graph having a triangle is at least 3-chromatic.

- A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.

Example



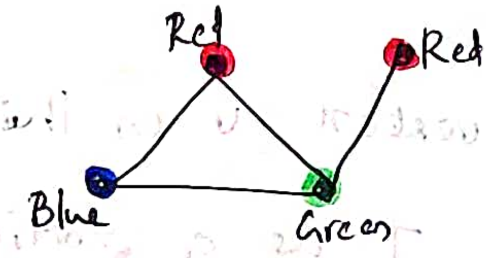
Different proper colouring of a graph G



→ 2 chromatic graph

- chromatic number 2

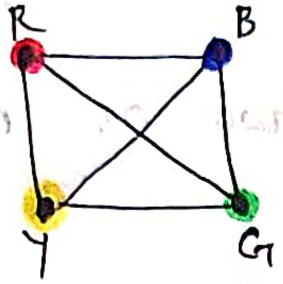
$\chi(G)=2$



- 3-chromatic graph

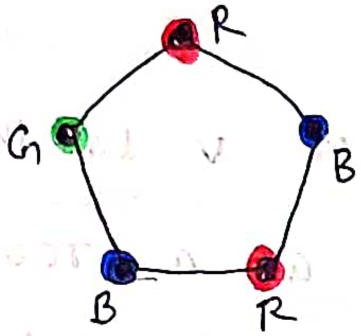
$\chi(G)=3$

→ 1-chromatic (isolated vertices)



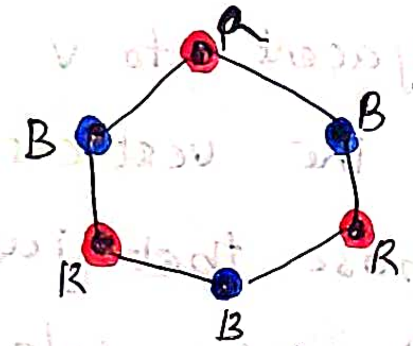
K_4 - 4-chromatic

$\chi(G) = 4$



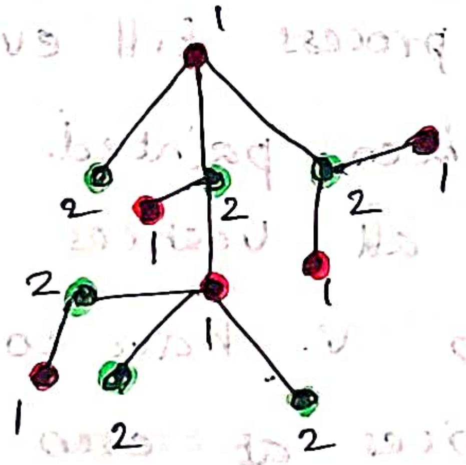
C_5 (odd cycle)

3-chromatic

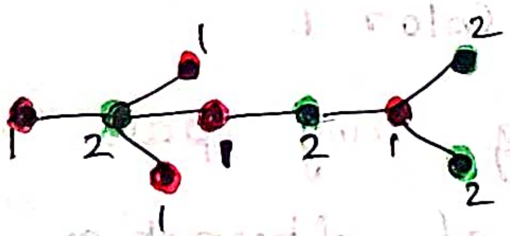


C_6 (even cycle)

2-chromatic



Tree - 2-chromatic



Tree - 2-chromatic.

Theorem

Every tree with two or more vertices is 2-chromatic.

Proof

Select any vertex v in the given tree T. Consider T as a rooted tree at vertex v. Paint v with color 1. Paint all vertices adjacent to v with color 2. Next paint the vertices adjacent to these (those that just have been coloured with 2) using color 1.

Continue this process till every vertex in T has been painted. Now in T we find that all vertices at odd distances from v have color 2 while v and vertices at even distances from v have color 1.

Now along any path in T the vertices are of alternating colors.

Since there is only one path between any two vertices in a tree, no two adjacent vertices have the same color. Thus T has been properly colored with two colors.

Theorem

A graph with at least one edge is 2-chromatic iff it has no circuits of odd length.

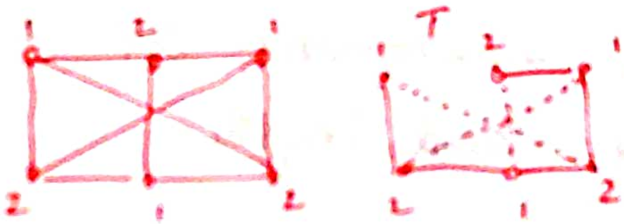
Proof

Let G be a graph with circuits of only even length. Consider a spanning tree T in G. Using the colouring procedure and by above theorem let us properly color T with two colours.

Now add the chords to T one by one. Since G had no circuits of odd length the end vertices of every chord being replaced and differently coloured in T. Thus G is colored with two colors with no adjacent vertices have the same color.

That is G is 2-chromatic.

Conversely, if G has circuits of odd length we would need at least three colors just for that circuit. Hence the theorem.



Theorem

If d_{max} is the maximum degree of the vertices in a graph G

chromatic number of $G \leq 1 + d_{max}$.

Notes

- Every 2-chromatic graph is bipartite.
- Every bipartite graph is 2-chromatic except the graph with 2 or more isolated vertices and with no-edge which is bipartite but 1-chromatic

Chromatic Polynomial $P_n(\lambda)$, $P_G(K)$

A given graph G of n vertices can be properly coloured in many different ways using a sufficiently large number of colours. This property of a graph is expressed by means of a polynomial. This polynomial is called the chromatic polynomial of G.

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph using λ or fewer colors.

Let c_i be the ways of properly coloring G using exactly i different colors. Since i colors can be chosen out of λ colors in $\binom{\lambda}{i}$ different ways there are $c_i \binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors.

Since i can be any +ve integer from 1 to n, the chromatic polynomial is the sum of these terms.

i.e.,
$$P_n(\lambda) = \sum_{i=1}^{n} c_i \binom{\lambda}{i} \qquad P_G(k)$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \dots + c_n \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!}$$

Theorem

A graph of n vertices is a complete graph iff its chromatic polynomial is,

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)$$

Proof:

With λ colors, there are λ different ways of coloring any selected vertex of a graph.

A second vertex can be colored

properly in exactly $\lambda-1$ ways the third in $\lambda-2$ ways the fourth in $\lambda-3$ ways and the nth in $\lambda-(n-1)$ ways iff every vertex is adjacent to every other. ie, if and only if the graph is complete.

Theorem

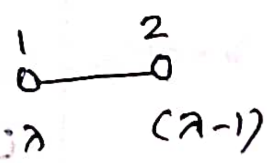
A graph of n vertices is a tree if and only if its chromatic polynomial is
$$P_n(\lambda) = \lambda(\lambda-1)^{n-1}.$$

Proof

Proof is by induction on number of vertices. Let $n=1$, ie, an isolated vertex then clearly the vertex can be coloured in λ ways with λ colors.

$$\therefore P_n(\lambda) = \lambda = \lambda(\lambda-1)^{1-1} = \lambda$$

when, $n=2$,



$$P_2(\lambda) = \lambda.(\lambda-1)$$
$$= \lambda(\lambda-1)^{2-1}$$

Theorem holds for $n = 1$ & 2

Now assum the theorem holds for $n=k$ vertices.

i.e., $P_n(\lambda) = \lambda(\lambda-1)^{k}$.

We will prove that the theorem is true for $k+1$ vertices.

Consider a tree with $k+1$ vertices

Every tree with $n \geq 2$ has minimum 2 pendant vertices.

Remove one of the pendant vertex then we left with a tree with k vertices. Then by hypothesis

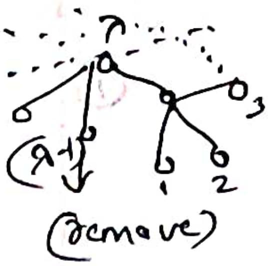
$$P_k(\lambda) = \lambda(\lambda-1)^{k-1}$$

After coloring all vertices in $\lambda(\lambda-1)^{k-1}$ ways attach the pendant vertex

Then the pendant vertex can be coloured ~~with~~ in $(\lambda-1)$ ways

Hence all vertices can be coloured in $\lambda(\lambda-1)^{k-1}(\lambda-1)$ ways in tree with $k+1$ vertices.

$$P_{k+1}(\lambda) = \lambda(\lambda-1)^{k-1}(\lambda-1)$$



$$= \lambda(\lambda-1)^{k}$$

$$= \lambda(\lambda-1)^{((k+1)-1)}$$

Hence the theorem holds for a tree with k+1 vertices also.

So by induction theorem is proved.

<u>Note:</u>

- A graph with n vertices and using n different <u>colors</u> can be properly colored in n! ways.

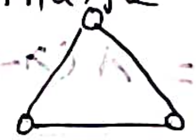
$$C_n = n!$$

- Suppose G is a graph with 2 vertices

2 minimum colours needed for proper colouring. $\therefore c_1 = 0$

(c_i - no. of ways graph can be colored using i colors)

Triangle

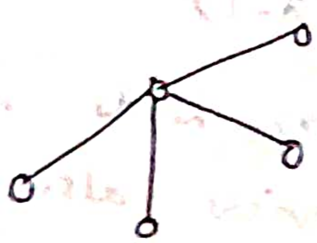


- require minimum 3 colours

$\therefore c_1=0,\ c_2=0\ \&\ c_3 = 3!$

① Find the chromatic polynomial of the following graph.

1)



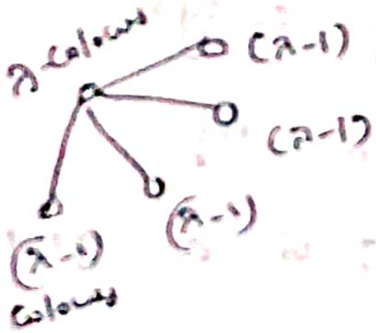
It's a tree with 5 vertices.

Hence the chromatic polynomial is,

$$P_5(\lambda) = \lambda(\lambda-1)^{5-1}$$

$$= \underline{\underline{\lambda(\lambda-1)^4}}$$

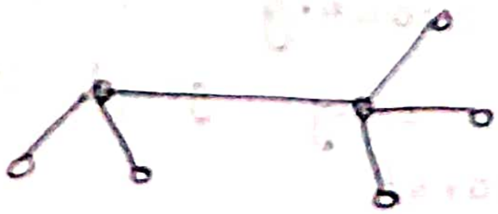
[or,



$\lambda \times (\lambda-1)(\lambda-1)(\lambda-1)(\lambda-1)$

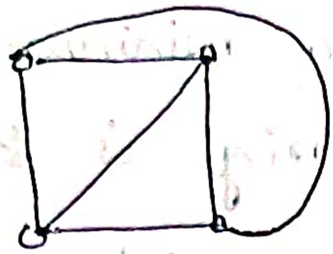
$= \lambda(\lambda-1)^4$]

2)



$P_7(\lambda) = \lambda(\lambda-1)^{7-1} = \underline{\underline{\lambda(\lambda-1)^6}}$

3) Find the chromatic polynomial of the graph

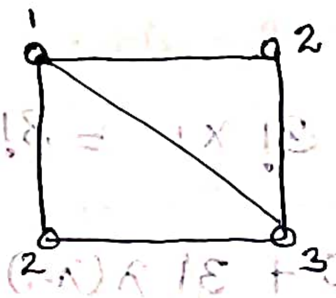


Given graph is K_4. a complete graph with 4 vertices

Hence, $P_4(\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-(4-1))$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

4) Find the chromatic polynomial of the graph



Here, the minimum colour required for a proper coloring is 3.

There are 4 vertices. Hence the polynomial is given by $P_4(\lambda) = \sum_{i=1}^{4} c_i \binom{\lambda}{i}$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{(\lambda)(\lambda-1)(\lambda-2)}{3!}$$
$$+ c_4 \frac{(\lambda)(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

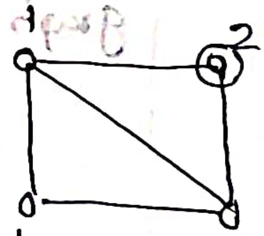
we have to find c_i

Here $c_1 = 0$, $c_2 = 0$ since minimum colour required for proper coloring is 3.

Also $c_4 = 4!$ since to color a graph with 4 vertices with 4 colors is in 4! different ways.

Now it remains to find c_3.

3 colors can be assigned to 3 vertices in 3! ways. The 4th vertex can be assigned with one of the 3 colors in each proper coloring.



4th vertex can be given color 2 similarly for all 3! coloring.

Hence $c_3 = 3! \times 1 = 3!$

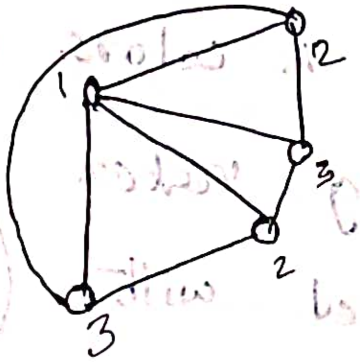
$$\therefore P_4(\lambda) = 0 + 0 + 3!\,\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 4!\,\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= \lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$= \lambda(\lambda-1)(\lambda-2)(1+\lambda-3)$$

$$= \lambda(\lambda-1)(\lambda-2)^2$$

5) Find the chromatic polynomial of the graph:

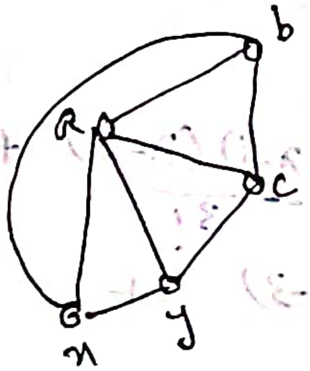


chromatic number is 3 for the graph

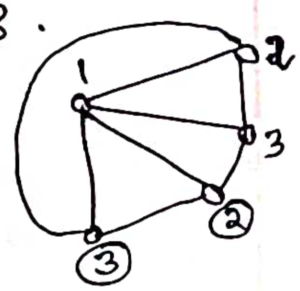
hence $C_1 = 0$, $C_2 = 0$

Also graph has five vertices $C_5 = 5!$

we need to obtain C_3 & C_4.

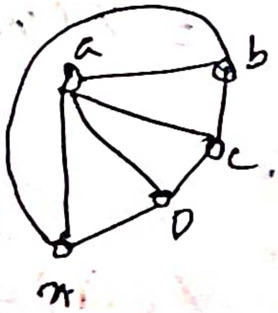


Consider the triangle abc with 3 colors triangle can be colored with 3! different ways. The other vertices x & y can be coloured with 2 colors from among the 3.

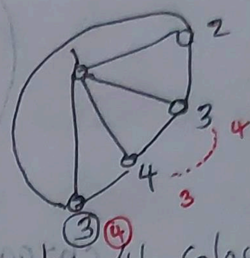


Hence, $C_3 = 3!$

Now to find C_4,

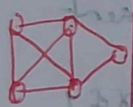


four of the vertices say a, b, c, & d can be colored with 4 colors in 4! ways and the remaining vertex ~~x can be colored with any of the 4 colours in 2 different ways.~~ & in each 4! colouring, can do in two different ways.



$$\therefore C_4 = 4! \times 2$$
$$= \cancel{12}\ 4 \times 3 \times 2 \times 2$$
$$= 48$$

(HW)



$$\therefore P_5(\lambda) = 0 + 0 + 3!\,\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 4! \times 2 \times \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 5! \times \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$= \lambda(\lambda-1)(\lambda-2)\left[1 + 2(\lambda-3) + (\lambda-3)(\lambda-4)\right]$$

$$= \lambda(\lambda-1)(\lambda-2)\left[\lambda^2 - 5\lambda + 7\right]$$

C_n - Cyclic graph

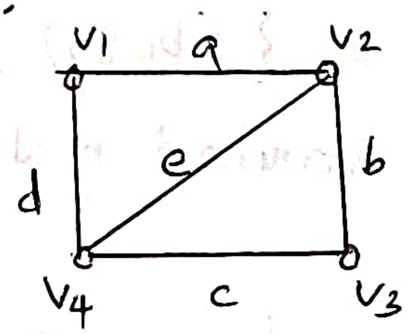
$\chi(C_n) = 2$, if n is even
$= 3$ if n is odd

Matching

A matching in a graph is a subset of edges in which no two edges are adjacent.

A single edge in a graph is obviously a matching.

Eg:



In the graph, {d, b} {a, c} & {e} are 3 matchings.

or $\{(v_1,v_4),(v_2,v_3)\}$, $\{(v_1,v_2),(v_3,v_4)\}$ & $\{(v_2,v_4)\}$

Maximal Matching

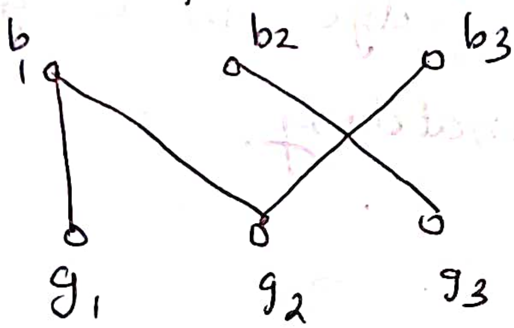
A maximal matching is a matching to which no edge in the graph can be added.

Eg: 1) In the above graph {d, b}, {a, c} {e} all are maximal matchings eg. 2

In a triangle any single edge is a maximal matching.

eg: 3

<u>Bipartite graph</u>



Matching
$\{(b_1, g_1), (b_2, g_3), (b_3, g_2)\}$

$\{(b_1, g_2), (b_2, g_3)\}$

both are maximal matching.

<u>Note:</u>

A graph may have many different maximal matching and of different size.

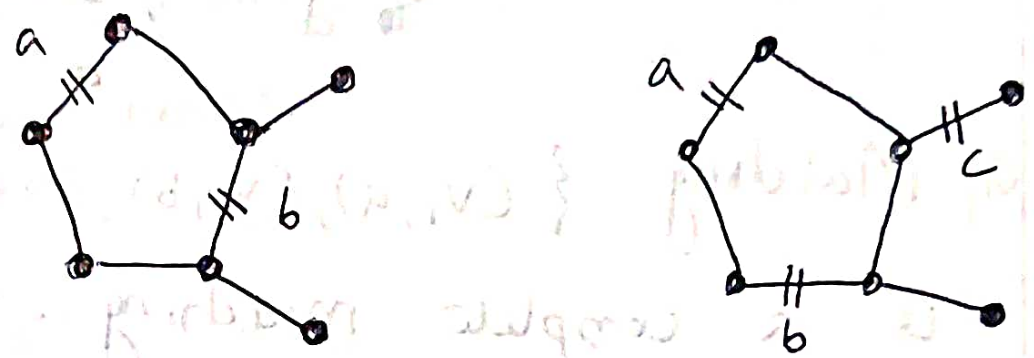
<u>Largest Maximal Matching</u> (Maximum matching)

The maximal matching with the largest number of edges are called largest maximal matching or Maximum matching.

Matching number of a graph

The number of edges in a largest maximal matching is called the matching number of the graph.

Eg:



{a, b} - Maximal matching

{a, b, c} is the largest maximal Matching

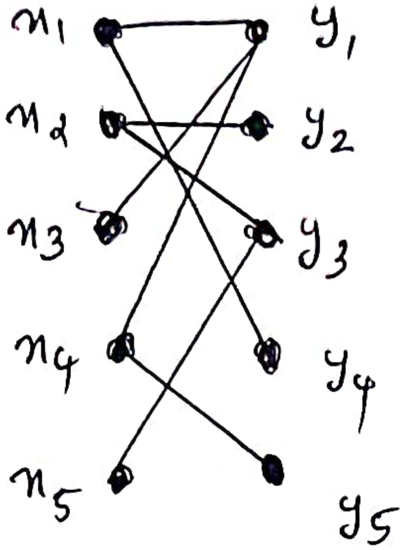
Matching number = 3

Complete Matching: (Perfect Matching)

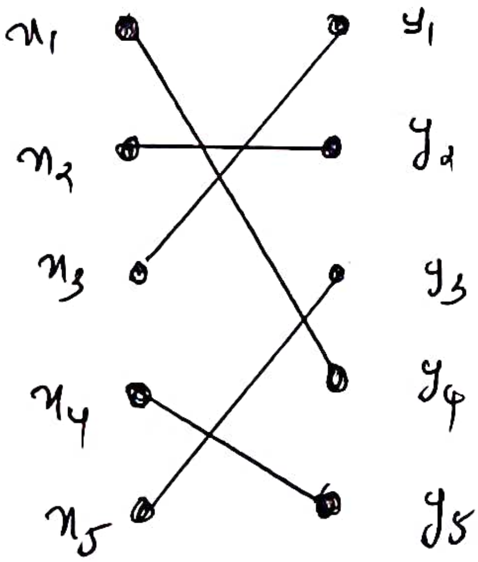
A complete matching in a graph is a matching that matches every vertex.

Clearly a complete matching (if exists) is a largest maximal matching, whereas the converse not necessarily true.

Qn. Find a maximum matching in the graph below. (Model Qn.)



Matching



$x_1 \to y_4$

$x_2 \to y_2$

$x_3 \to y_1$

$x_4 \to y_5$

$x_5 \to y_3$

which is a perfect matching.

(Assign the pendant vertices with the first available matching)

Coverings:

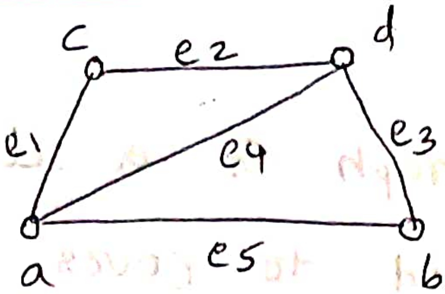
In a graph G a set g of edges is said to cover G if every vertex in G is incident on at least one edge in g.

A set of edges that covers a graph G is said to be an edge covering, a covering subgraph or simply a covering of G.

Note:

- A graph G is trivially its own covering.
- A spanning tree in a connected graph or (a spanning forest in an unconnected graph is a covering
- A Hamiltonian circuit (if it exists in a graph is a covering

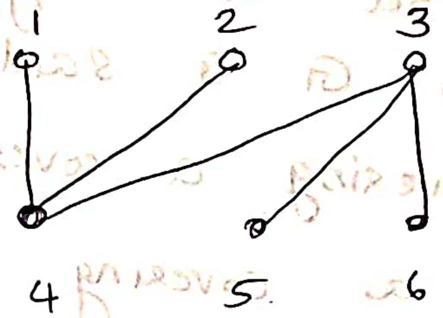
Eg: 1)



$\{e_1, e_3\}$ is an edge-covering

$\{e_1, e_5\}$ is not an edge-covering

Eg: 2



$\{(1,4), (2,4), (3,5), (3,6)\}$ is an edge covering.

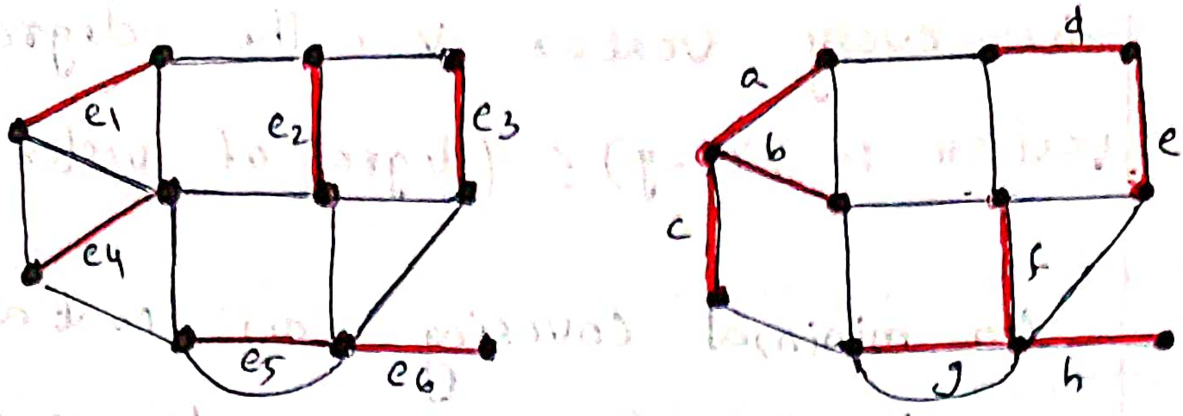
Minimal Covering

A covering from which no edge can be removed without destroying its ability to cover the graph

Covering number

The number of edges in a minimal covering of the smallest size is called the covering number of the graph.

$\{e_1, e_2, e_3, e_4, e_5, e_6\}$ $\{a, b, c, d, e, f, g, h\}$



Graph and two of its minimal covering.

Note:

- A covering exists for a graph if and only if the graph has no isolated vertex.
- A covering of an n-vertex graph will have at least $\lceil \frac{n}{2} \rceil$ edges.
- Every pendant edge in a graph is included in every covering of the graph
- Every covering contains a minimal covering.
- If we denote the remaining edges of a graph by $(G-g)$, the set of edges g is a covering iff

for every vertex v, the degree of vertex in $(G - g) \leq$ (degree of vertex v in G) -1

- No minimal covering can contain a circuit, for we can always remove an edge from a circuit without leaving any of the vertices in the circuit uncovered. Therefore a minimal covering of an n-vertex graph can contain no more than n-1 edges.
- A graph in general has many minimal covering and they may be of different sizes.

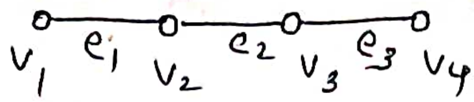
Theorem

A covering g of a graph is minimal if and only if g contains no paths of length three or more.

Proof

Suppose that a covering g

contains a path of length three and it is

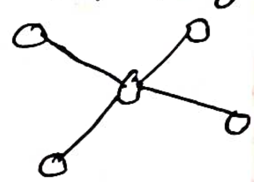


$v_1 e_1 v_2 e_2 v_3 e_3 v_4$

then edge e_2 can be removed without leaving its end vertices v_2 and v_3 uncovered. Therefore g is not a minimal covering.

Conversely if a covering g contains no path of length three or more all its components must be star graphs.

From a stargraph no edge can be removed without leaving a vertex uncovered.



That is g must be a minimal covering.

Four color Problem (Four color conjecture)

Consider the proper coloring of regions in a planar graph. Just as in coloring of vertices and edges, the regions of a planar graph are said to be properly colored if no two contiguous or adjacent regions have the same color. (Two regions are said to be adjacent if they have a common edge between them)

The proper coloring of regions is called map coloring.

The Four colour conjecture is that every map (i.e, a planar graph) can be properly colored with 4 colors.

No one has yet been able to either prove the theorem or come up with a map that requires more than four colors.

The four color conjucture can be restated as follows.

Every planar graph has a chromatic number of four or less.

Five - color Theorem

Every planar map can be properly colored with five colors.